Complex scaling for the Dirichlet Laplacian in a domain with asymptotically cylindrical end ⋆

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Abstract

We develop the complex scaling method for the Dirichlet Laplacian in a domain with asymptotically cylindrical end. We define resonances as discrete eigenvalues of non-selfadjoint operators, obtained as deformations of the selfadjoint Dirichlet Laplacian Δ by means of the complex scaling. The resonances are identified with the poles of the resolvent matrix elements $((\Delta-\mu)^{-1}F,G)$ meromorphic continuation in μ across the essential spectrum of Δ, where F and G are elements of an explicitly given set of partial analytic vectors. It turns out that the Dirichlet Laplacian has no singular continuous spectrum, and its eigenvalues can accumulate only at threshold values of the spectral parameter.

1 Introduction

We consider the Dirichlet Laplacian in a domain $\mathcal{G}\subset\mathbb{R}^{n+1}$ with asymptotically cylindrical end. By the asymptotically cylindrical end we mean any unbounded domain obtained as the transformation of the semi-cylinder $(0,\infty)\times\Omega$ by some suitable diffeomorphism $\varkappa$ that does not distort the space at infinity; here $\Omega\subset\mathbb{R}^n$ is a bounded simply connected domain. Outside of some compact set the domain $\mathcal{G}$ coincides with its asymptotically cylindrical end, and the boundary of $\mathcal{G}$ is smooth; see Fig. 1.

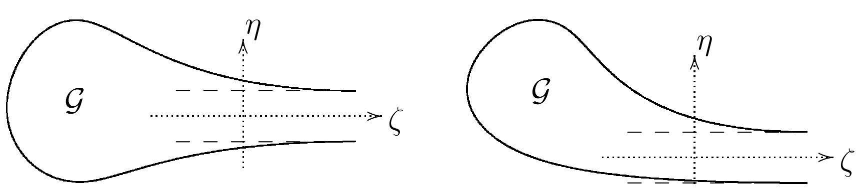

Fig. 1. Domain $\mathcal{G}\subset\mathbb{R}^{n+1}$ with asymptotically cylindrical end.

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Equivalently, the asymptotically cylindrical end can be viewed as a Riemannian manifold $((0,\infty)\times\Omega, \varkappa^*\mathsf{e})$ isometrically embedded into the space $\mathbb{R}^{n+1}$ with the Euclidean metric e. Then the pullback $\varkappa^*\mathsf{e}$ of e by the diffeomorphism $\varkappa$ converges in a certain sense to e as the axial coordinate x of the semi-cylinder $(0,\infty)\times\Omega$ goes to infinity. Usually, when studying the Laplacian in this kind of geometry, one imposes some assumptions on the rate of convergence of the metric $\varkappa^*\mathsf{e}$ to its limit at infinity, e.g. [4,7,8,9,19,20,21]. Our goal is to study the case of arbitrarily slow convergence. With this aim in mind we invoke the complex scaling method. This method allows us to use an assumption on the analytic regularity of the diffeomorphism $\varkappa$ with respect to the axial coordinate x as a substitution for the assumptions on the rate of convergence of the metric $\varkappa^*\mathsf{e}$ at infinity.

The complex scaling method has a long tradition in the spectral and scattering theories, see e.g. [1,3,5,6,10,12,17,18,23,24,25,27] and references therein. Nevertheless the complex scaling has not been used in this geometric setting before. We first develop the complex scaling method per se. We introduce a deformation of the Dirichlet Laplacian by means of the complex scaling, and study the spectrum of the deformed operator. Here we essentially rely on ideas of the Aguilar-Balslev-Combes-Simon theory of resonances [1,3,5,6,10,23], and most of all on the approach to the complex scaling introduced in [12]. As might be expected from known situations, the deformation by means of the complex scaling separates the essential spectrum of the Dirichlet Laplacian Δ from the non-threshold eigenvalues, rotating the rays of the essential spectrum about the thresholds. For locating the essential spectrum we employ the theory of elliptic boundary value problems [14,15]. We characterize the spectrum of the deformed Dirichlet Laplacian, establishing an analog of the celebrated Aguilar-Balslev-Combes theorem. This becomes possible due to the fact that on some sufficiently large (dense) set of partial analytic vectors $\mathcal{A}$ the resolvent matrix elements $((\Delta-\mu)^{-1}F,G)$ with $F,G\in\mathcal{A}$ are intimately connected with resolvent matrix elements of the operator Δ deformed by means of the complex scaling. It turns out that for all $F,G\in\mathcal{A}$ the function $\mu\mapsto((\Delta-\mu)^{-1}F,G)$ can be meromorphically continued from the physical sheet $\mathbb{C}\setminus(0,\infty)$ across the essential spectrum of Δ to a Riemann surface. This implies that the Dirichlet Laplacian has no singular continuous spectrum. The non-real poles of the resolvent matrix elements are identified with the resonances of the Dirichlet Laplacian, that are introduced as the non-real eigenvalues of the deformed operator. The real poles correspond to the eigenvalues of the Dirichlet Laplacian. The eigenvalues embedded to the essential spectrum are known to be very unstable, under rather weak perturbations they become resonances [2,10].

Under our assumptions on the domain $\mathcal{G}$, accumulations of isolated and embedded eigenvalues of the Dirichlet Laplacian may occur e.g. [7]. The Aguilar-Balslev-Combes theorem implies that the non-threshold eigenvalues of the Dirichlet Laplacian are of finite multiplicities, and can accumulate only at the

thresholds. In a companion paper we will apply the complex scaling in order to establish the following properties of the Dirichlet Laplacian: a) the non-threshold eigenfunctions are of exponential decay at infinity; b) the eigenvalues are of finite multiplicity; c) the eigenvalues can accumulate at the thresholds only from below. Note that in [7] decay of eigenfunctions and eigenvalue accumulations of the Dirichlet and Neumann Laplacians are studied by another method and under an assumption on the form of the metric and on the rate of its convergence at infinity.

We complete the introduction with a description of the structure of this paper. In Section 2 we formulate and discuss our results. The subsequent sections are devoted to the proof. Section 3 presents our approach to the complex scaling. In Section 4 we localize the essential spectrum of the Dirichlet Laplacian deformed by means of the complex scaling. In Section 5 we consider a suitable set $\mathcal{A}$ of partial analytic vectors. Finally, in Section 6 we construct the resolvent matrix elements meromorphic continuation and complete the proof of our results.

2 Statement and discussion of results

Let (x, y) and (ζ, η) be two systems of the Cartesian coordinates in $\mathbb{R}^{n+1}$, $n \geq 1$, such that $x, \zeta \in \mathbb{R}$, while $y = (y_1, \dots, y_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ are in $\mathbb{R}^n$. Let $\partial_x = \frac{d}{dx}$, $\partial_{y_m} = \frac{d}{dy_m}$, and $\partial_\zeta = \frac{d}{d\zeta}$, $\partial_{\eta_m} = \frac{d}{d\eta_m}$.

Consider a closed bounded simply connected domain $\Omega \subset \mathbb{R}^n$, and the semi-cylinder $\Pi = \mathbb{R}_+ \times \Omega$, where $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. We say that $\mathcal{C} \subset \mathbb{R}^{n+1}$ is an asymptotically cylindrical end, if there exists a diffeomorphism

$$\Pi \ni (x, y) \mapsto \varkappa(x, y) = (\zeta, \eta) \in \mathcal{C}, \tag{2.1}$$

such that the elements $\varkappa'_{\ell m}(x, \cdot)$ of its Jacobian matrix $\varkappa'$ tend to the Kronecker delta $\delta_{\ell m}$ in the space $C^\infty(\Omega)$ as $x \to +\infty$.

Let $\mathcal{G}$ be a closed domain in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\mathcal{G}$. We suppose that the set $\{(\zeta, \eta) \in \mathcal{G} : \zeta \leq 0\}$ is bounded, and the set $\{(\zeta, \eta) \in \mathcal{G} : \zeta > 0\}$ is the asymptotically cylindrical end $\mathcal{C}$, cf. Fig. 1. Introduce the notation $\nabla_{\zeta\eta} = (\partial_\zeta, \partial_{\eta_1}, \dots, \partial_{\eta_n})^\top$. In the domain $\mathcal{G}$ we consider the Dirichlet Laplacian $\Delta = -\nabla_{\zeta\eta} \cdot \nabla_{\zeta\eta}$, which is initially defined on the set $C_0^\infty(\mathcal{G})$ of all smooth functions with compact supports in $\mathcal{G} \setminus \partial\mathcal{G}$.

In this paper we use the complex scaling $x \mapsto x + \lambda v(x)$. Here λ is a scaling parameter, and $v(x)$ is a smooth function possessing the properties:

$$v(x) = 0 \text{ for } x \le R \text{ with a sufficiently large } R > 0, \tag{2.2}$$
$$0 \le v'(x) \le 1 \text{ for all } x \in \mathbb{R}, \tag{2.3}$$
$$v'(x) = 1 \text{ for } x \ge \widetilde{R} > R, \tag{2.4}$$

where $v'(x) = \partial_x v(x)$, and $\widetilde{R}$ is arbitrary. For all real $\lambda \in (-1, 1)$ the function $\mathbb{R}_+ \ni x \mapsto x + \lambda v(x)$ is invertible, and $\kappa_\lambda(x, y) = (x + \lambda v(x), y)$ is a selfdiffeomorphism of the semi-cylinder Π. Therefore

$$\vartheta_\lambda(\zeta, \eta) = \begin{cases} \varkappa \circ \kappa_\lambda \circ \varkappa^{-1}(\zeta, \eta) & \text{for} \quad (\zeta, \eta) \in \mathcal{C}, \\ (\zeta, \eta) & \text{for} \quad (\zeta, \eta) \in \mathcal{G} \setminus \mathcal{C}, \end{cases} \tag{2.5}$$

is a selfdiffeomorphism of the domain $\mathcal{G}$. In other words, ϑ_λ with $\lambda \in (-1, 1)$ is a scaling of the end $\mathcal{C}$ along the curvilinear coordinate x.

Let $(\vartheta'_\lambda)^\top$ be the transpose of the Jacobian matrix ϑ'_λ. Then $\mathsf{h}_\lambda = (\vartheta'_\lambda)^\top \vartheta'_\lambda$ is the matrix coordinate representation of a Riemannian metric h_λ on $\mathcal{G}$, and

$${}^\lambda\Delta = -\Big(\det \mathsf{h}_\lambda\Big)^{-1/2} \nabla_{\zeta\eta} \cdot \Big(\det \mathsf{h}_\lambda\Big)^{1/2} \mathsf{h}_\lambda^{-1} \nabla_{\zeta\eta} \tag{2.6}$$

is the Laplace-Beltrami operator on the Riemannian manifold $(\mathcal{G}, \mathsf{h}_\lambda)$. As the parameter R in (2.2) increases, the equalities $\vartheta_\lambda(\zeta, \eta) = (\zeta, \eta)$, $\mathsf{h}_\lambda = \mathrm{Id}$, where Id is the $(n+1) \times (n+1)$ identity matrix, and the equality ${}^\lambda\Delta = \Delta$, become valid on a larger and larger subset of $\mathcal{G}$. In the case $\lambda = 0$ the scaling is not applied, and ${}^0\Delta \equiv \Delta$.

In order to consider complex values of the scaling parameter λ, we make additional assumptions on the partial analytic regularity of the diffeomorphism $\varkappa$ in (2.1):

i. the function $\mathbb{R}_+ \ni x \mapsto \varkappa(x, \cdot) \in C^\infty(\Omega)$ has an analytic continuation from $\mathbb{R}_+$ to some sector
$$\mathbb{S}_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha < \pi/4\}; \tag{2.7}$$

ii. the elements $\varkappa'_{\ell m}(z, \cdot)$ of the Jacobian matrix $\varkappa'$ tend to the Kronecker delta $\delta_{\ell m}$ in the space $C^\infty(\Omega)$ uniformly in $z \in \mathbb{S}_\alpha$ as $z \to \infty$.

For instance, the assumptions *i*,*ii* are satisfied for the following ends $\mathcal{C} \subset \mathbb{R}^2$:

$$\mathcal{C} = \{(\zeta, \eta) \in \mathbb{R}^2 : (\zeta, \eta) = (x, y + \log(x+2)), \ x \in \mathbb{R}_+, \ y \in [0, 1]\},$$

$$\mathcal{C} = \Big\{(\zeta, \eta) \in \mathbb{R}^2 : \zeta = \int_0^x \varphi(t)dt, \ \eta = y\psi(x), \ x \in \mathbb{R}_+, \ y \in [0, 1]\Big\},$$

where as $\varphi(x)$ and $\psi(x)$ we can take the functions 1, $1 + e^{-x}$, $1 + (x+1)^{-s}$ with $s > 0$, $1 + 1/\log(x+2)$, $1 + 1/\log(1 + \log(x+2))$, and so on.

It turns out that the assumptions *i, ii* together with (2.2) and (2.3) lead to the analyticity of the coefficients of the differential operator (2.6) with respect to the scaling parameter λ in the disc

$$\mathcal{D}_\alpha = \{\lambda \in \mathbb{C} : |\lambda| < \sin\alpha\}. \tag{2.8}$$

We take the equality (2.6) as definition of the differential operator ${}^\lambda\Delta$ for all $\lambda \in \mathcal{D}_\alpha$. The operator ${}^\lambda\Delta$ in $L^2(\mathcal{G})$, initially defined on the set $C_0^\infty(\mathcal{G})$, is closable. Here $L^2(\mathcal{G})$ is the Hilbert space with the usual norm $\|u; L^2(\mathcal{G})\| = \left(\int_\mathcal{G} |u(\zeta,\eta)|^2\, d\zeta\, d\eta\right)^{1/2}$. The closure of ${}^\lambda\Delta$, denoted by the same symbol ${}^\lambda\Delta$, is an unbounded operator in $L^2(\mathcal{G})$, which is nonselfadjoint for $\lambda \neq 0$. However the Dirichlet Laplacian ${}^0\Delta \equiv \Delta$ is selfadjoint and positive. We consider the operator ${}^\lambda\Delta$ with $\lambda \in \mathcal{D}_\alpha$ as a deformation of Δ by means of the complex scaling. The essential spectrum $\sigma_{ess}({}^\lambda\Delta)$ of ${}^\lambda\Delta$ depends only on the behaviour of the matrix h_λ outside any compact region of $\mathcal{G}$. In order to control $\sigma_{ess}({}^\lambda\Delta)$ we imposed the condition (2.4). Before formulating our results, we introduce partial analytic vectors.

Let $L^2(\Omega)$ be the space with the norm $(\int_\Omega |f(y)|^2\, dy)^{1/2}$. Consider the algebra $\mathscr{E}$ of all entire functions $\mathbb{C} \ni z \mapsto f(z,\cdot) \in C^\infty(\Omega)$ with the following property: in any sector $|\Im z| \le (1-\epsilon)\Re z$ with $\epsilon > 0$ the value $\|f(z,\cdot); L^2(\Omega)\|$ decays faster than any inverse power of $\Re z$ as $\Re z \to +\infty$. Examples of functions $f \in \mathscr{E}$ are $f(z,y) = e^{-\gamma z^2} P(z,y)$, where $\gamma > 0$ and $P(z,y)$ is an arbitrary polynomial in z with coefficients in $C^\infty(\Omega)$. We say that $F \in L^2(\mathcal{G})$ is a partial analytic vector, if $F \circ \varkappa(x,y) = f(x,y)$ for some $f \in \mathscr{E}$ and all $(x,y) \in \Pi$. The set of all partial analytic vectors is denoted by $\mathcal{A}$. Later on we will show that $\mathcal{A}$ is dense in the space $L^2(\mathcal{G})$.

Theorem 2.1 *Let $\nu_1 < \nu_2 < \dots$ be the distinct eigenvalues of the selfadjoint Dirichlet Laplacian $\Delta_\Omega = -\partial_{\eta_1}^2 - \dots - \partial_{\eta_n}^2$ in the space $L^2(\Omega)$. Let the scaling parameter λ be in the disc* (2.8)*, where α is the same as in the conditions i, ii on the diffeomorphism $\varkappa$. Assume that the deformation ${}^\lambda\Delta$ of the Dirichlet Laplacian Δ is constructed with a smooth scaling function $v(x)$ satisfying the conditions* (2.2)–(2.4)*. Then the following assertions are valid.*

1. *The spectrum $\sigma({}^\lambda\Delta)$ of the operator ${}^\lambda\Delta$ is independent of the choice of $v(x)$.*
2. *μ is a point of the essential spectrum $\sigma_{ess}({}^\lambda\Delta)$, if and only if*

$$\mu = \nu_j \ \text{or}\ \arg(\mu - \nu_j) = -2\arg(1+\lambda) \ \text{for some}\ j \in \mathbb{N}. \tag{2.9}$$

3. *$\sigma({}^\lambda\Delta) = \sigma_{ess}({}^\lambda\Delta) \cup \sigma_d({}^\lambda\Delta)$, where $\sigma_d({}^\lambda\Delta)$ is the discrete spectrum of ${}^\lambda\Delta$.*
4. *Let $\mu \in \sigma_d({}^\lambda\Delta)$. As λ changes continuously in the disc $\mathcal{D}_\alpha$, the point μ remains in $\sigma_d({}^\lambda\Delta)$ as long as $\mu \in \mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$.*
5. *Let $(\cdot,\cdot)$ stand for the inner product in $L^2(\mathcal{G})$. For any $F, G \in \mathcal{A}$ the analytic function $\mathbb{C}\setminus\overline{\mathbb{R}_+} \ni \mu \mapsto \big((\Delta-\mu)^{-1}F, G\big)$ has a meromorphic continuation to*

the set $\mathbb{C}\setminus\sigma_{ess}({}^{\lambda}\Delta)$. Moreover, μ is a pole of the meromorphic continuation with some $F,G\in\mathcal{A}$, if and only if $\mu\in\sigma_d({}^{\lambda}\Delta)$.

6. *A point $\mu\in\mathbb{R}$, such that $\mu\neq\nu_j$ for all $j\in\mathbb{N}$, is an eigenvalue of the Dirichlet Laplacian Δ, if and only if $\mu\in\sigma_d({}^{\lambda}\Delta)$ with $\Im\lambda\neq 0$.*
7. *The Dirichlet Laplacian Δ has no singular continuous spectrum.*

A similar result for the stationary Schrödinger operator in $\mathbb{R}^n$ is known as the Aguilar-Balslev-Combes theorem, see e.g. [10,23] and references therein.

The spectral portrait of the operator ${}^{\lambda}\Delta$ is depicted on Fig. 2. We say that the numbers $\nu_1,\nu_2,\dots$ are threshold values of the spectral parameter μ, or thresholds for short. As the parameter λ varies, the ray $\arg(\mu-\nu_j)=-2\arg(1+\lambda)$

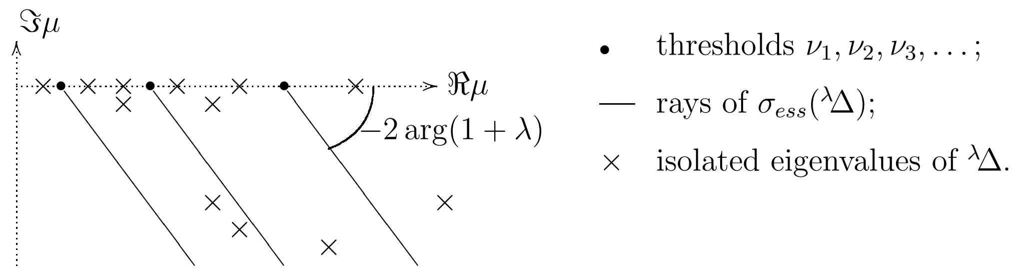

Fig. 2. Spectral portrait of the deformed Dirichlet Laplacian ${}^{\lambda}\Delta$, $\Im\lambda>0$.

of the essential spectrum $\sigma_{ess}({}^{\lambda}\Delta)$ rotates about the threshold ν_j, and sweeps the sector $|\arg(\mu-\nu_j)|<2\alpha$. By the assertion *4* the eigenvalues of ${}^{\lambda}\Delta$ outside of the sector $|\arg(\mu-\nu_1)|<2\alpha$ do not change, hence they are the discrete eigenvalues of the selfadjoint Dirichlet Laplacian Δ. All other discrete eigenvalues of ${}^{\lambda}\Delta$ belong to the sector $|\arg(\mu-\nu_1)|<2\alpha$. As λ varies, they remain unchanged until they are covered by one of the rotating rays of the essential spectrum. Conversely, new eigenvalues can be uncovered by the rotating rays. In the case $\Im\lambda\geq 0$ (resp. $\Im\lambda\leq 0$) the operator ${}^{\lambda}\Delta$ cannot have eigenvalues in the half-plane $\Im\mu>0$ (resp. $\Im\mu<0$). Indeed, by the assertion *4* a number μ with $\Im\mu>0$ is an eigenvalue of ${}^{\lambda}\Delta$ with $\Im\lambda\geq 0$, if and only if μ is an eigenvalue of Δ, but the Dirichlet Laplacian Δ cannot have non-real eigenvalues as a selfadjoint operator. Further, by the assertion *6* the real eigenvalues $\mu\in\sigma_d({}^{\lambda}\Delta)$ survive for $\lambda=0$: the eigenvalues $\mu<\nu_1$ become the discrete eigenvalues of Δ, while the eigenvalues $\mu>\nu_1$ become the embedded non-threshold eigenvalues of Δ. In view of the fact that any non-threshold point μ can be separated from $\sigma_{ess}({}^{\lambda}\Delta)$ by a small variation of $\arg(1+\lambda)$, the set $\sigma_d({}^{\lambda}\Delta)$ (and therefore the set of all eigenvalues of Δ) has no accumulation points, except possibly for the thresholds $\nu_1,\nu_2,\dots$. Suitable examples of accumulating eigenvalues can be found e.g. in [7]. By definition, all discrete non-real eigenvalues of ${}^{\lambda}\Delta$ are resonances of the Dirichlet Laplacian. By the assertion *5* the resonances are characterized by the pair $\{\Delta,\mathcal{A}\}$. They are identified with the complex poles of the meromorphic continuation to a Riemann surface of all resolvent matrix

elements $\mu \mapsto \big((\Delta-\mu)^{-1}F, G\big)$ with $F, G \in \mathcal{A}$. Readers might have noticed a certain analogy between the situation we described above and the one known from the theory of resonances for N-body problem e.g. [10,12].

3 Complex scaling

In this section we show that the differential operator (2.6) is well defined for complex values of the scaling parameter λ, and obtain some estimates on its coefficients.

Consider the asymptotically cylindrical end $\mathcal{C}$ as a Riemannian manifold endowed with the Euclidean metric e. We will use the coordinates (ζ, η) in $\mathcal{G}$ and (x, y) in Π, and identify the Riemannian metrics on Π and $\mathcal{G}$ with their matrix coordinate representations. Let $\mathsf{g} = \varkappa^*\mathsf{e}$ be the pullback of e by the diffeomorphism $\varkappa$ in (2.1). Then the matrix $\mathsf{g} = [\mathsf{g}_{\ell m}]_{\ell,m=1}^{n+1}$ is given by the equality $\mathsf{g} = (\varkappa')^\top \varkappa'$, where $(\varkappa')^\top$ is the transpose of the Jacobian $\varkappa'$. Since the diffeomorphism $\varkappa$ satisfies the assumptions *i,ii* of Section 2, we conclude that the metric matrix elements

$$\mathbb{S}_\alpha \ni z \mapsto \mathsf{g}_{\ell m}(z, \cdot) \in C^\infty(\Omega) \tag{3.1}$$

are analytic functions. Moreover, $\mathsf{g}_{\ell m}(z, \cdot)$ tends to the Kronecker delta $\delta_{\ell m}$ in the space $C^\infty(\Omega)$ uniformly in $z \in \mathbb{S}_\alpha$ as $z \to \infty$ or, equivalently, we have

$$\Big|\partial_y^q(\mathsf{g}_{\ell m}(z, y) - \delta_{\ell m})\Big| \le C_q(|z|) \to 0 \text{ as } z \to \infty,\ z \in \mathbb{S}_\alpha,\ y \in \Omega,\ |q| \ge 0; \tag{3.2}$$

here $\partial_y^q = \partial_{y_1}^{q_1}\partial_{y_2}^{q_2}\dots\partial_{y_n}^{q_n}$ with a multiindex $q = (q_1, \dots, q_n)$, and $|q| = \sum q_j$.

Consider the selfdiffeomorphism $\kappa_\lambda(x, y) = (x + \lambda v(x), y)$ of the semi-cylinder Π, where $\lambda \in (-1, 1)$, and $v(x)$ is a smooth scaling function with the properties (2.2)–(2.4). We define the metric $\mathsf{g}_\lambda = \kappa_\lambda^*\mathsf{g}$ on Π as the pullback of the metric g by κ_λ. As a result we get Riemannian manifolds $(\Pi, \mathsf{g}_\lambda)$ parametrized by $\lambda \in (-1, 1)$. For the matrix representation of g_λ we deduce the expression

$$\mathsf{g}_\lambda(x, y) = \operatorname{diag}\{1 + \lambda v'(x), \mathrm{Id}\}\, \mathsf{g}(x + \lambda v(x), y)\, \operatorname{diag}\{1 + \lambda v'(x), \mathrm{Id}\}\,, \tag{3.3}$$

where Id stands for the $n \times n$-identity matrix, and $\operatorname{diag}\{1 + \lambda v'(x), \mathrm{Id}\}$ is the Jacobian of κ_λ. The matrix g_λ is invertible for all $\lambda \in (-1, 1)$. For brevity of notations we do not indicate the dependence of v, κ_λ, and g_λ on the large parameter R in (2.2).

Now we wish to consider complex values of the scaling parameter λ. We suppose that λ is in the complex disc $\mathcal{D}_\alpha$, where α is the same as in our assumptions on the partial analytic regularity of the diffeomorphism $\varkappa$; cf. (2.7), (2.8).

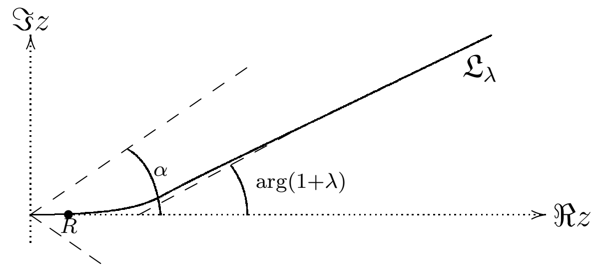

Fig. 3. The curve $\mathfrak{L}_\lambda$ for complex values of λ.

The function $\mathbb{R}_+ \ni x \mapsto x + \lambda v(x)$ is invertible, and the curve $\mathfrak{L}_\lambda = \{z \in \mathbb{C} : z = x + \lambda v(x), x > 0\}$ lies in the sector $\mathbb{S}_\alpha$, see Fig. 3. We define the matrix g_λ for all non-real λ in the disc by the equality (3.3), where $\mathsf{g}(x+\lambda v(x), y)$ stands for the value of the analytic in $z \in \mathbb{S}_\alpha$ function $\mathsf{g}(z, y)$ at $z = x+\lambda v(x)$. By analyticity in λ we conclude that g_λ is a complex symmetric matrix, the Schwartz reflection principle gives $\overline{\mathsf{g}_\lambda} = \mathsf{g}_{\overline{\lambda}}$, where the overline stands for the complex conjugation. If $\lambda \in \mathcal{D}_\alpha$ is non-real, then the matrix g_λ does not correspond to a Riemannian metric.

Let us show that the matrix $\mathsf{g}_\lambda(x, y)$ is invertible for all $(x, y) \in \Pi$ and $\lambda \in \mathcal{D}_\alpha$. Evidently, $v(x) = 0$ and $\mathsf{g}_\lambda^{-1}(x, y) = \mathsf{g}^{-1}(x, y)$ for all $x < R$. On the other hand, for all $x \geq R > 0$ we have $|x + \lambda v(x)| \geq R$. Therefore for a sufficiently large $R > 0$ the matrix $\mathsf{g}(x+\lambda v(x), y)$ with $x \geq R$ is little different from the identity matrix due to (3.2). This implies invertibility of the matrix $\mathsf{g}(x+\lambda v(x), y)$ for all $\lambda \in \mathcal{D}_\alpha$ and $(x, y) \in \Pi$. By analyticity the equality

$$\mathsf{g}_\lambda^{-1}(x, y) = \operatorname{diag}\left\{\frac{1}{1 + \lambda v'(x)}, \operatorname{Id}\right\} \mathsf{g}^{-1}(x + \lambda v(x), y) \operatorname{diag}\left\{\frac{1}{1 + \lambda v'(x)}, \operatorname{Id}\right\} \tag{3.4}$$

extends from real to all $\lambda \in \mathcal{D}_\alpha$. Clearly, the derivatives $\partial_x^p \partial_y^q \mathsf{g}_\lambda^{-1}$ are analytic functions of $\lambda \in \mathcal{D}_\alpha$. On the next step we obtain some estimates on these derivatives.

Let $\partial_z = \frac{1}{2}(\partial_{\Re z} - i\partial_{\Im z})$ be the complex derivative. Due to analyticity in z we have $|\partial_z^p \partial_y^q \mathsf{g}_{\ell m}(z, y)| \leq C_p \max_{\tilde{z} \in \mathcal{O}(z)} |\partial_y^q \mathsf{g}_{\ell m}(\tilde{z}, y)|$, where $\mathcal{O}(z)$ is a small circle centered at z. Therefore the conditions (3.1) and (3.2) on the metric g lead to the uniform in $y \in \Omega$ and $\{z \in \mathbb{C} : |\arg(z - R)| < \alpha\}$ estimates

$$\left|\partial_z^p \partial_y^q (\mathsf{g}_{\ell m}(z, y) - \delta_{\ell m})\right| \leq \mathrm{c}_{pq}(R) \to 0 \text{ as } R \to +\infty, \quad \ell, m = 1, \dots, n+1.$$

This together with (2.3) and (3.4) implies that

$$\left\|\partial_x^p \partial_y^q \Big(\mathsf{g}_\lambda^{-1}(x, y) - \operatorname{diag}\big\{(1 + \lambda)^{-2}, \operatorname{Id}\big\}\Big)\right\| \to 0 \text{ as } x \to +\infty,\ y \in \Omega,\ \lambda \in \mathcal{D}_\alpha, \tag{3.5}$$

and estimate

$$\left\|\partial_x^p \partial_y^q \Big(\mathsf{g}_\lambda^{-1}(x, y) - \operatorname{diag}\big\{(1 + \lambda v'(x))^{-2}, \operatorname{Id}\big\}\Big)\right\| \leq \mathrm{C}_{pq}(R) \tag{3.6}$$

holds uniformly in $(x,y) \in [R,\infty) \times \Omega$ and $\lambda \in \mathcal{D}_\alpha$, where $\|\cdot\|$ is the matrix norm $\|A\| = \max_{\ell m} |a_{\ell m}|$, and $p + |q| \geq 0$. The constants $\mathrm{C}_{pq}(R)$ in (3.6) tend to zero as $R \to +\infty$; recall that v and g_λ both depend on the parameter R.

We define the complex scaling ϑ_λ for all λ in the disc $\mathcal{D}_\alpha$ by the equality (2.5), where $\varkappa \circ \kappa_\lambda(x,y)$ is the value of the analytic in $z \in \mathbb{S}_\alpha$ function $\varkappa(z,y)$ at the point $z = x + \lambda v(x)$. Consider the matrix $\mathsf{h}_\lambda = (\vartheta'_\lambda)^\top \vartheta'_\lambda$. It is clear that for all $(\zeta,\eta) \in \mathcal{G} \setminus \mathcal{C}$ the matrix $\mathsf{h}_\lambda(\zeta,\eta)$ coincides with the $(n+1) \times (n+1)$-identity. For all real $\lambda \in \mathcal{D}_\alpha$ the Riemannian geometry gives the identity

$$\mathsf{g}_\lambda(x,y) = \Big(\varkappa'(x,y)\Big)^\top \Big(\mathsf{h}_\lambda \circ \varkappa(x,y)\Big)\varkappa'(x,y), \quad (x,y) \in \Pi, \tag{3.7}$$

where $\mathsf{g}_\lambda = \varkappa^* \mathsf{h}_\lambda$ is the pullback of the corresponding metric h_λ on $\mathcal{C}$ by the diffeomorphism $\varkappa$. The Jacobian $\varkappa'$ is an invertible and independent of λ matrix, and the matrix g_λ is invertible and analytic in $\lambda \in \mathcal{D}_\alpha$. Therefore the matrix h_λ is invertible and analytic in $\lambda \in \mathcal{D}_\alpha$ due to (3.7). By analyticity in λ we conclude that $\mathsf{h}_\lambda(\zeta,\eta)$ is a complex symmetric matrix, the Schwartz reflection principle gives $\overline{\mathsf{h}_\lambda} = \mathsf{h}_{\overline{\lambda}}$.

Differentiating the equality (3.7), we see that the derivatives $\partial_\zeta^p \partial_\eta^q \mathsf{h}_\lambda$ and $\partial_\zeta^p \partial_\eta^q \mathsf{h}_\lambda^{-1}$ are analytic in $\lambda \in \mathcal{D}_\alpha$. Moreover, from (3.5) and (3.6) together with our assumptions on the diffeomorphism $\varkappa$ we have

$$\begin{gathered}
\Big\| \partial_\zeta^p \partial_\eta^q \Big(\mathsf{h}_\lambda^{-1}(\zeta,\eta) - \operatorname{diag}\Big\{ \Big(1+\lambda\Big)^{-2}, \mathrm{Id} \Big\} \Big) \Big\| \to 0 \text{ as } \zeta \to +\infty, \\
\Big\| \partial_\zeta^p \partial_\eta^q \Big(\mathsf{h}_\lambda^{-1}(\zeta,\eta) - \operatorname{diag}\Big\{ \Big(1+\lambda v'(\zeta,\eta)\Big)^{-2}, \mathrm{Id} \Big\} \Big) \Big\| \leq c(R), \\
\text{where } p + |q| \leq 1, \text{ and } \ c(R) \to 0 \text{ as } R \to +\infty.
\end{gathered} \tag{3.8}$$

Here $(\zeta,\eta) \in \mathcal{C}$ and $v'(\zeta,\eta) \equiv v' \circ \varkappa^{-1}(\zeta,\eta)$. We extend v' from $\mathcal{C}$ to $\mathcal{G}$ by zero, then the estimate (3.8) extends to all $(\zeta,\eta) \in \mathcal{G}$. The constant $c(R)$ in (3.8) is independent of $\lambda \in \mathcal{D}_\alpha$ and $(\zeta,\eta) \in \mathcal{G}$. Note that the matrix h_λ with $\lambda \neq 0$ depends on the large parameter R, however we do not indicate this in notations. Now we see that the differential operator (2.6) is well defined for all λ in the disc $\mathcal{D}_\alpha$, and its coefficients are subjected to the estimate (3.8).

4 Localization of the essential spectrum

Introduce the the Sobolev space $\mathring{H}^2(\mathcal{G})$ as the completion of the set $C_0^\infty(\mathcal{G})$ with respect to the norm

$$\|u; \mathring{H}^2(\mathcal{G})\| = \|\Delta u; L^2(\mathcal{G})\| + \|u; L^2(\mathcal{G})\|. \tag{4.1}$$

In this section we prove the following proposition.

Proposition 4.1 *1. The unbounded operator ${}^\lambda\Delta$ in $L^2(\mathcal{G})$ with the domain $\mathring{H}^2(\mathcal{G})$ is closed.*

2. The continuous operator ${}^\lambda\Delta - \mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$ is not Fredholm, if and only if $\mu \in \mathbb{C}$ and $\lambda \in \mathcal{D}_\alpha$ meet the condition (2.9).

Recall that μ is said to be a point of the essential spectrum $\sigma_{ess}({}^\lambda\Delta)$ of the closed unbounded operator ${}^\lambda\Delta$ in $L^2(\mathcal{G})$ with the domain $\mathring{H}^2(\mathcal{G})$, if the continuous operator ${}^\lambda\Delta - \mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$ is not Fredholm (a linear continuous operator between two Banach spaces is Fredholm, if its kernel and cokernel are finite dimensional, and its range is closed). Thus the assertion *2* of Theorem 2.1 is a direct consequence of Proposition 4.1.

The proof of Proposition 4.1 is essentially based on methods of the theory of elliptic non-homogeneous boundary value problems [14,15,16]. The proof is preceded by the following lemma.

Lemma 4.2 *The operator ${}^\lambda\Delta$ is strongly elliptic for all $\lambda \in \mathcal{D}_\alpha$.*

PROOF. Outside of the support of the scaling function v the operator ${}^\lambda\Delta$ coincides with the strongly elliptic operator Δ. Hence we only need to check the strong ellipticity of ${}^\lambda\Delta$ inside the image of the set $[R, \infty) \times \Omega$ under the diffeomorphism $\varkappa$.

Consider the operator

$$ {}^\lambda\Delta_{\mathsf{g}} = -\Big(\det \mathsf{g}_\lambda\Big)^{-1/2} \nabla_{xy} \cdot \Big(\det \mathsf{g}_\lambda\Big)^{1/2} \mathsf{g}_\lambda^{-1} \nabla_{xy}, \quad \lambda \in \mathcal{D}_\alpha, \tag{4.2} $$

where g_λ is the matrix (3.3) and $\nabla_{xy} = (\partial_x, \partial_{y_1} \dots \partial_{y_n})^\top$. From (3.7) and (2.6) it is easily seen that ${}^\lambda\Delta_{\mathsf{g}}$ is the operator ${}^\lambda\Delta$ written in the curvilinear coordinates (x, y) inside the end $\mathcal{C}$; i.e. ${}^\lambda\Delta u = \Big({}^\lambda\Delta_{\mathsf{g}}(u \circ \varkappa)\Big) \circ \varkappa^{-1}$ for all $u \in C_0^\infty(\mathcal{C})$. Since the strong ellipticity is preserved under the diffeomorphisms, it suffices to show that the operator ${}^\lambda\Delta_{\mathsf{g}}$ is strongly elliptic on $[R, \infty) \times \Omega \subset \Pi$.

By virtue of the bounds $|\lambda| < \sin\alpha$, $0 < \alpha < \pi/4$, and $0 \le v'(x) \le 1$ we have

$$ \Re\Big(\xi \cdot \operatorname{diag}\Big\{(1 + \lambda v'(x))^{-2}, \mathrm{Id}\Big\}\xi\Big) \ge (\cos 2\alpha)|\xi|^2/4 \tag{4.3} $$

for all $x \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^{n+1}$. Now we make use of the estimates (3.6) on the matrix $\mathsf{g}_\lambda^{-1}(x, y)$. Since R is sufficiently large, the constant $\mathrm{C}_{00}(R)$ in (3.6) meets the estimate $(n+1)^2\mathrm{C}_{00}(R) < (\cos 2\alpha)/4$. This together with (4.3)

implies the uniform in $\lambda \in \mathcal{D}_\alpha$ and $(x,y) \in [R,\infty)\times\Omega$ estimates

$$\begin{aligned}\Re\Big(\xi\cdot \mathsf{g}_\lambda^{-1}(x,y)\xi\Big) &\geq \Re\Big(\xi\cdot \operatorname{diag}\Big\{(1+\lambda v'(x))^{-2}, \mathrm{Id}\Big\}\xi\Big)\\ &-\Big|\xi\cdot\Big(\mathsf{g}_\lambda^{-1}(x,y) - \operatorname{diag}\Big\{(1+\lambda)^{-2},\mathrm{Id}\Big\}\Big)\xi\Big|\\ &\geq \Big((\cos 2\alpha)/4 - (n+1)^2 \mathrm{C}_{00}(R)\Big)|\xi|^2\end{aligned}$$

on the principal symbol of ${}^\lambda\Delta_{\mathsf{g}}$.

Proof of Proposition 4.1 We will rely on the following lemma due to Peetre, see e.g. [16, Lemma 5.1], [15, Lemma 3.4.1] or [22]:

Let $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces, where $\mathcal{X}$ is compactly embedded into $\mathcal{Z}$. Furthermore, let $\mathcal{L}$ be a linear continuous operator from $\mathcal{X}$ to $\mathcal{Y}$. Then the next two assertions are equivalent: (i) the range of $\mathcal{L}$ is closed in $\mathcal{Y}$ and $\dim\ker\mathcal{L}<\infty$*, (ii) there exists a constant C, such that*

$$\|u;\mathcal{X}\| \leq C(\|\mathcal{L}u;\mathcal{Y}\| + \|u;\mathcal{Z}\|) \quad \forall u\in\mathcal{X}. \tag{4.4}$$

Below we assume that μ and λ does not meet the condition (2.9) and establish the coercive estimate

$$\|u;H^2(\mathcal{G})\| \leq C(\|({}^\lambda\Delta-\mu)u;L^2(\mathcal{G})\| + \|\mathsf{w}u;L^2(\mathcal{G})\|) \quad \forall u\in \mathring{H}^2(\mathcal{G}) \tag{4.5}$$

for the operator ${}^\lambda\Delta-\mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$. Here $\mathsf{w}\in C^\infty(\mathcal{G})$ is a positive rapidly decreasing at infinity weight, such that the embedding of $\mathring{H}^2(\mathcal{G})$ into the weighted space $L^2(\mathcal{G};\mathsf{w})$ with the norm $\|\mathsf{w}\cdot;L^2(\mathcal{G})\|$ is compact. The coercive estimate (4.5) is an estimate of type (4.4).

As is well-known, a strongly elliptic operator and the Dirichlet boundary condition set up a regular elliptic boundary value problem, e.g. [16]. Solutions of a regular elliptic boundary value problem satisfy local coercive estimates, e.g. [16] or [15]. As a consequence of Lemma 4.2 we get the local coercive estimate

$$\|\rho_T u;H^2(\mathcal{G})\| \leq C(\|\varrho_T({}^\lambda\Delta-\mu)u;L^2(\mathcal{G})\| + \|\varrho_T u;L^2(\mathcal{G})\|) \quad \forall u\in\mathring{H}^2(\mathcal{G}). \tag{4.6}$$

Here $\rho_T,\varrho_T\in C^\infty(\mathcal{G})$ are compactly supported cutoff functions, such that $\rho_T(\zeta,\eta)=1$ for $|\zeta|<T+1$ and $\varrho_T\rho_T=\rho_T$, where T is a large fixed number.

Let $\chi_T\in C^\infty(\mathcal{G})$ be another cutoff function, such that $\chi_T(\zeta,\eta)=1$ for $|\zeta|>T$ and $\chi_T(\zeta,\eta)=0$ for $|\zeta|<T-1$. On the next step we establish the estimate (4.4) with u replaced by $\chi_T u$. We will do it in the coordinates $(x,y)\in\Pi$.

Let $L^2(\mathbb{R}\times\Omega)$ be the space of functions in the infinite cylinder $\mathbb{R}\times\Omega$ with the norm $\Big(\int_{\mathbb{R}}\|\mathsf{u}(x);L^2(\Omega)\|^2\,dx\Big)^{1/2}$. Introduce the Sobolev space $\mathring{H}^2(\mathbb{R}\times\Omega)$

of functions with zero Dirichlet data on $\mathbb{R}\times\partial\Omega$ as the completion of the set $C_0^\infty(\mathbb{R}\times\Omega)$ with respect to the norm

$$\|\mathsf{u};\mathring{H}^2(\mathbb{R}\times\Omega)\|=\Big(\sum_{p+|q|\le 2}\|\partial_x^p\partial_y^q\mathsf{u};L^2(\mathbb{R}\times\Omega)\|^2\Big)^{1/2}.$$

Let $\mathsf{u}=(\chi_T u)\circ\varkappa$, where $\varkappa$ is the diffeomorphism (2.1). Due to our assumptions on $\varkappa$ the estimates $0<\epsilon\le\det\varkappa'(x,y)\le 1/\epsilon$ hold uniformly in $(x,y)\in\Pi$. Hence for some independent of $u\in C_0^\infty(\mathcal{G})$ constants c_1, c_2, and c_3 we have

$$\begin{aligned}&\|\chi_T u;\mathring{H}^2(\mathcal{G})\|=\|\Delta(\chi_T u);L^2(\mathcal{G})\|+\|\chi_T u;L^2(\mathcal{G})\|\\&\quad\le c_1(\|{}^0\Delta_{\mathsf{g}}\mathsf{u};L^2(\mathbb{R}\times\Omega)\|+\|\mathsf{u};L^2(\mathbb{R}\times\Omega)\|)\le c_2\|\mathsf{u};\mathring{H}^2(\mathbb{R}\times\Omega)\|,\\&\|({}^\lambda\Delta_{\mathsf{g}}-\mu)\mathsf{u};L^2(\mathbb{R}\times\Omega)\|\le c_3\|({}^\lambda\Delta-\mu)\chi_T u;L^2(\mathcal{G})\|.\end{aligned}\tag{4.7}$$

Here the functions u and ${}^\lambda\Delta_{\mathsf{g}}\mathsf{u}=({}^\lambda\Delta(\chi_T u))\circ\varkappa$ are extended from Π to the infinite cylinder $\mathbb{R}\times\Omega$ by zero, and $\|{}^0\Delta_{\mathsf{g}}\mathsf{u};L^2(\mathbb{R}\times\Omega)\|\le C\|\mathsf{u};\mathring{H}^2(\mathbb{R}\times\Omega)\|$ because the coefficients of the Laplacian ${}^0\Delta_{\mathsf{g}}$ are bounded, cf. (4.2) and (3.6). As T is large, the function u is supported in a small neighborhood of infinity. Due to the stabilization condition (3.5) on g_λ^{-1} the coefficients of the differential operator ${}^\lambda\Delta_{\mathsf{g}}-\Delta_\Omega+(1+\lambda)^{-2}\partial_x^2$ are small on the support of u. As a result we get the estimate

$$\Big\|\Big({}^\lambda\Delta_{\mathsf{g}}-\Delta_\Omega+(1+\lambda)^{-2}\partial_x^2\Big)\mathsf{u};L^2(\mathbb{R}\times\Omega)\Big\|\le\epsilon\|\mathsf{u};\mathring{H}^2(\mathbb{R}\times\Omega)\|,\tag{4.8}$$

where ϵ is small and independent of $u\in C_0^\infty(\mathcal{G})$; moreover, $\epsilon\to 0$ as $T\to+\infty$.

Consider the continuous operator

$$\Delta_\Omega-(1+\lambda)^{-2}\partial_x^2-\mu:\mathring{H}^2(\mathbb{R}\times\Omega)\to L^2(\mathbb{R}\times\Omega).\tag{4.9}$$

Applying the Fourier transform $\mathscr{F}_{x\mapsto\tau}$ we pass from the operator (4.9) to the Dirichlet Laplacian $\Delta_\Omega+(1+\lambda)^{-2}\tau^2-\mu$ in $L^2(\Omega)$. Since μ and λ does not meet the condition (2.9), the spectral parameter $\mu-(1+\lambda)^{-2}\tau^2$ is outside of the spectrum $\{\nu_j\}_{j=1}^\infty$ of Δ_Ω for all $\tau\in\mathbb{R}$. Then a known argument, see e.g. [15, Theorem 5.2.2], [14, Theorem 2.4.1], implies that the operator (4.9) realizes an isomorphism. In particular the estimate

$$\|\mathsf{u};\mathring{H}^2(\mathbb{R}\times\Omega)\|\le \mathrm{c}\Big\|\Big(\Delta_\Omega-(1+\lambda)^{-2}\partial_x^2-\mu\Big)\mathsf{u};L^2(\mathbb{R}\times\Omega)\Big\|\tag{4.10}$$

is valid with an independent of $\mathsf{u}\in\mathring{H}^2(\mathbb{R}\times\Omega)$ constant c; in order to make the paper selfcontained, we establish the estimate (4.10) in Lemma 4.3 below the proof. As a consequence of (4.10) and (4.8) we have

$$\begin{aligned}(1-\epsilon\mathrm{c})\|\mathsf{u};\mathring{H}^2(\mathbb{R}\times\Omega)\|&\le\mathrm{c}\Big\|\Big(\Delta_\Omega-(1+\lambda)^{-2}\partial_x^2-\mu\Big)\mathsf{u};L^2(\mathbb{R}\times\Omega)\Big\|\\&-\mathrm{c}\Big\|\Big({}^\lambda\Delta_{\mathsf{g}}-\Delta_\Omega+(1+\lambda)^{-2}\partial_x^2\Big)\mathsf{u};L^2(\mathbb{R}\times\Omega)\Big\|\le\mathrm{c}\|({}^\lambda\Delta_{\mathsf{g}}-\mu)\mathsf{u};L^2(\mathbb{R}\times\Omega)\|.\end{aligned}$$

If T is sufficiently large, then $\epsilon \mathrm{c} < 1$. This together with (4.7) gives

$$\|\chi_T u; \mathring{H}^2(\mathcal{G})\| \leq C\|({}^\lambda\Delta - \mu)\chi_T u; L^2(\mathcal{G})\|, \tag{4.11}$$

where the constant $C = \mathrm{c}(1 - \epsilon\mathrm{c})^{-1}c_2c_3$ is independent of $u \in C_0^\infty(\mathcal{G})$. By continuity the estimate (4.11) extends to all $u \in \mathring{H}^2(\mathcal{G})$.

Now we combine (4.11) with (4.6), and arrive at the estimates

$$\begin{aligned}\|u; \mathring{H}^2(\mathcal{G})\| \leq \|\chi_T u; \mathring{H}^2(\mathcal{G})\| + \|\rho_T u; \mathring{H}^2(\mathcal{G})\| \leq C(\|\chi_T({}^\lambda\Delta - \mu)u; L^2(\mathcal{G})\| \\ + \|[{}^\lambda\Delta - \mu, \chi_T]u; L^2(\mathcal{G})\| + \|\varrho({}^\lambda\Delta - \mu)u; L^2(\mathcal{G})\| + \|\varrho u; L^2(\mathcal{G})\|) \\ \leq C(\|({}^\lambda\Delta - \mu)u; L^2(\mathcal{G})\| + \|\varrho u; L^2(\mathcal{G})\|).\end{aligned} \tag{4.12}$$

Here we used that $\rho_T = 1$ on the support of the commutator $[{}^\lambda\Delta - \mu, \chi_T]$, and hence

$$\|[{}^\lambda\Delta - \mu, \chi_T]u; L^2(\mathcal{G})\| \leq C\|\rho_T u; \mathring{H}^2(\mathcal{G})\|.$$

For an arbitrary positive weight $\mathsf{w} \in C^\infty(\mathcal{G})$ we have

$$\|\varrho u; L^2(\mathcal{G})\| \leq C\|\mathsf{w}u; L^2(\mathcal{G})\|$$

with an independent of $u \in \mathring{H}^2(\mathcal{G})$ constant C. Thus the estimate (4.4) is a direct consequence of (4.12). By the Peetre's lemma we conclude that the range of the continuous operator ${}^\lambda\Delta - \mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$ is closed and the kernel is finite-dimensional. The estimate (4.4) with $\mathsf{w} \equiv 1$ shows that $\mathring{H}^2(\mathcal{G})$ is the domain of the closed operator ${}^\lambda\Delta$ in $L^2(\mathcal{G})$, which proves the assertion *1.*

In order to see that the cokernel $\operatorname{coker}({}^\lambda\Delta - \mu) = \ker({}^\lambda\Delta^* - \overline{\mu})$ of the continuous operator ${}^\lambda\Delta - \mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$ is finite-dimensional (if μ and λ does not meet the condition (2.9)), we obtain the coercive estimate

$$\|u; \mathring{H}^2(\mathcal{G})\| \leq C(\|({}^\lambda\Delta^* - \overline{\mu})u; L^2(\mathcal{G})\| + \|\mathsf{w}u; L^2(\mathcal{G})\|) \tag{4.13}$$

for the adjoint ${}^\lambda\Delta^*$ of the unbounded operator ${}^\lambda\Delta$ in $L^2(\mathcal{G})$, and apply the Peetre's lemma. The proof of the estimate (4.13) is similar to the proof of (4.4), we omit it.

We have proved that the operator ${}^\lambda\Delta - \mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$ is Fredholm, if λ and μ do not satisfy the condition (2.9). Now we assume that λ and μ satisfy the condition (2.9) for some j, and show that the operator ${}^\lambda\Delta - \mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$ is not Fredholm.

Let χ be a smooth cutoff function on the real line, such that $\chi(x) = 1$ for

$|x-3| \le 1$ and $\chi(x)=0$ for $|x-3| \ge 2$. Consider the functions

$$\mathsf{u}_\ell(x,\mathrm{y}) = \chi(x/\ell)\exp\Big(i(1+\lambda)\sqrt{\mu-\nu_j x}\Big)\Phi(\mathrm{y}), \quad (x,\mathrm{y})\in\mathbb{R}\times\Omega, \tag{4.14}$$

where Φ is an eigenfunction of Δ_Ω, corresponding to the eigenvalue ν_j. The exponent in (4.14) is an oscillating function of x. Straightforward calculation shows that

$$\Big\|\Big(\Delta_\Omega-(1+\lambda)^{-2}\partial_x^2-\mu\Big)\mathsf{u}_\ell; L^2(\mathbb{R}\times\Omega)\Big\| \le C, \quad \|\mathsf{u}_\ell; \mathring{H}^2(\mathbb{R}\times\Omega)\| \to \infty \tag{4.15}$$

as $\ell\to+\infty$. Similarly to (4.8) we conclude that

$$\Big\|\Big({}^\lambda\Delta_{\mathsf{g}}-\Delta_\Omega+(1+\lambda)^{-2}\partial_x^2\Big)\mathsf{u}_\ell; L^2(\mathbb{R}\times\Omega)\Big\| \le \epsilon_\ell\|\mathsf{u}_\ell; \mathring{H}^2(\mathbb{R}\times\Omega)\|, \tag{4.16}$$

where $\epsilon_\ell\to 0$ as $\ell\to+\infty$. Let the functions $u_\ell=\mathsf{u}_\ell\circ\varkappa^{-1}$ be extended from $\mathcal{C}$ to $\mathcal{G}$ by zero. If the operator ${}^\lambda\Delta-\mu:\mathring{H}^2(\mathcal{G})\to L^2(\mathcal{G})$ is Fredholm, then by the Peetre's lemma the estimate (4.5) holds with any weight w, such that $\mathring{H}^2(\mathcal{G})\hookrightarrow L^2(\mathcal{G};\mathsf{w})$ is a compact embedding. Without loss of generality we can assume that $\|\mathsf{w}u_\ell; L^2(\mathcal{G})\|\le C$ for all $\ell\ge 1$. After the change of variables $(\zeta,\eta)\mapsto(x,y)$ the estimate (4.5) implies

$$\|\mathsf{u}_\ell; \mathring{H}^2(\mathbb{R}\times\Omega)\| \le C(\|({}^\lambda\Delta_{\mathsf{g}}-\mu)\mathsf{u}_\ell; L^2(\mathbb{R}\times\Omega)\|+1),$$

where ${}^\lambda\Delta_{\mathsf{g}}\mathsf{u}_\ell=({}^\lambda\Delta u_\ell)\circ\varkappa$ is extended from Π to $\mathbb{R}\times\Omega$ by zero. This together with (4.16) justifies the estimate

$$\|\mathsf{u}_\ell; \mathring{H}^2(\mathbb{R}\times\Omega)\| \le C\Big(\Big\|\Big(\Delta_\Omega-(1+\lambda)^{-2}\partial_x^2-\mu\Big)\mathsf{u}_\ell; L^2(\mathbb{R}\times\Omega)\Big\|+1\Big),$$

which contradicts (4.15).□

Lemma 4.3 *Assume that* $|\lambda|<2^{-1/2}$ *and* $\mu\in\mathbb{C}$ *do not meet the condition* (2.9). *Then the estimate* (4.10) *holds with a constant* c *independent of* $\mathsf{u}\in\mathring{H}^2(\mathbb{R}\times\Omega)$.

PROOF. The differential operator $\Delta_\Omega-(1+\lambda)^{-2}\partial_x^2-\mu$ with $|\lambda|<2^{-1/2}$ is strongly elliptic, cf. (4.3). Therefore the local coercive estimate

$$\begin{aligned}\|\varrho\mathsf{u}; \mathring{H}^2(\mathbb{R}\times\Omega)\|^2 \le c\Big(\Big\|\varsigma\Big(\Delta_\Omega-(1+\lambda)^{-2}\partial_x^2-\mu\Big)\mathsf{u}; L^2(\mathbb{R}\times\Omega)\Big\|^2 \\ +\|\varsigma\mathsf{u}; L^2(\mathbb{R}\times\Omega)\|^2\Big)\end{aligned} \tag{4.17}$$

is valid, where ϱ and ς are smooth functions of the variable x with compact supports, and such that $\varrho\not\equiv 0$, $\varrho\varsigma=\varrho$.

Introduce the Sobolev space $\mathring{H}^\ell(\Omega)$ as the completion of the set $C_0^\infty(\Omega)$ with respect to the norm $\sum_{|q|\le\ell}\|\partial_y^q\Psi; L^2(\Omega)\|$. We substitute $\mathsf{u}(x,y)=e^{i\tau x}\Psi(y)$

with $\tau \in \mathbb{R}$ and $\Psi \in \mathring{H}^2(\Omega)$ into (4.17). After simple manipulations we arrive at the estimate

$$\sum_{p=0}^{2} |\tau|^{2p} \|\Psi; \mathring{H}^{2-p}(\Omega)\|^2 \leq C\Big(\Big\| \Big(\Delta_\Omega + (1+\lambda)^{-2}\tau^2 - \mu\Big)\Psi; L^2(\Omega)\Big\|^2 + \|\Psi; L^2(\Omega)\|^2 \Big), \tag{4.18}$$

where the constant C depends on ϱ and ς, but not on τ or Ψ.

We also have the elliptic coercive estimate

$$\|\Psi; \mathring{H}^2(\Omega)\|^2 \leq c\Big(\Big\| \Big(\Delta_\Omega + (1+\lambda)^{-2}\tau^2 - \mu\Big)\Psi; L^2(\Omega)\Big\|^2 + \|\Psi; L^2(\Omega)\|^2 \Big) \tag{4.19}$$

for the Dirichlet Laplacian in Ω. Since λ and μ do not meet the condition (2.9), the distance d between the ray $\{\mu - (1+\lambda)^{-2}\tau^2 : \tau \in \mathbb{R}\}$ and the spectrum $\{\nu_j\}_{j=1}^\infty$ of the selfadjoint operator Δ_Ω is positive. Therefore

$$\|\Psi; L^2(\Omega)\|^2 \leq d^{-2} \Big\| \Big(\Delta_\Omega + (1+\lambda)^{-2}\tau^2 - \mu\Big)\Psi; L^2(\Omega)\Big\|^2.$$

The estimate (4.19) takes the form

$$\|\Psi; \mathring{H}^2(\Omega)\|^2 \leq (c + d^{-2}) \Big\| \Big(\Delta_\Omega + (1+\lambda)^{-2}\tau^2 - \mu\Big)\Psi; L^2(\Omega)\Big\|^2. \tag{4.20}$$

If $|\tau| > r$ with sufficiently large $r > 0$, then the last term in (4.18) can be neglected. This together with (4.20) justifies the estimate

$$\sum_{p=0}^{2} |\tau|^{2p} \|\Psi; \mathring{H}^{2-p}(\Omega)\|^2 \leq C \Big\| \Big(\Delta_\Omega + (1+\lambda)^{-2}\tau^2 - \mu\Big)\Psi; L^2(\Omega)\Big\|^2 \tag{4.21}$$

for all $\tau \in \mathbb{R}$ and some independent of Ψ and τ constant C.

Let $\Psi(\tau)$ be the Fourier transform $\int_{\mathbb{R}} e^{i\tau x} \mathsf{u}(x)\,dx$ of $\mathsf{u} \in \mathring{H}^2(\mathbb{R} \times \Omega)$. Then $(-i\tau)^p \Psi(\tau)$ is the Fourier transform of $\partial_x^p \mathsf{u}(x)$. The Parseval equality gives

$$\int_{\mathbb{R}} |\tau|^{2p} \|\Psi(\tau); \mathring{H}^{2-p}(\Omega)\|^2\, d\tau = 2\pi \int_{\mathbb{R}} \|\partial_x^p \mathsf{u}(x); \mathring{H}^{2-p}(\Omega)\|^2\, dx.$$

Integrating (4.21) with respect to $\tau \in \mathbb{R}$, we obtain the estimate (4.10).

5 Partial analytic vectors

Consider the set of partial analytic vectors $\mathcal{A}$, introduced in Section 2. Recall that $F \in L^2(\mathcal{G})$ is in the set $\mathcal{A}$, if $F \circ \varkappa(x,y) = f(x,y)$ for some $f \in \mathscr{E}$

and all $(x,y) \in \Pi$. Here $\mathscr{E}$ is the algebra of all entire functions $\mathbb{C} \ni z \mapsto f(z,\cdot) \in C^\infty(\Omega)$, such that in any sector $|\Im z| \le (1-\epsilon)\Re z$ with $\epsilon > 0$ the value $\|f(z,\cdot); L^2(\Omega)\|$ decays faster than any inverse power of $\Re z$ as $\Re z \to +\infty$.

For $F \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ we define the function $F \circ \vartheta_\lambda$ in $\mathcal{G}$ by the equalities $F \circ \vartheta_\lambda(\zeta,\eta) = F(\zeta,\eta)$ for $(\zeta,\eta) \in \mathcal{G} \setminus \mathcal{C}$, and

$$F \circ \vartheta_\lambda \circ \varkappa(x,y) = f(x + \lambda v(x), y) \text{ for } (x,y) \in \Pi. \tag{5.1}$$

Here $f(x+\lambda v(x),\cdot)$ is the value of the corresponding to F entire function $f \in \mathscr{E}$ at the point $z = x + \lambda v(x)$.

Proposition 5.1 *Let v be a smooth function satisfying* (2.2)–(2.4)*. Then*

1. *For any $F \in \mathcal{A}$, $\lambda \mapsto F \circ \vartheta_\lambda$ is an $L^2(\mathcal{G})$-valued analytic function in the disc $|\lambda| < 2^{-1/2}$;*
2. *For any λ in the disc $|\lambda| < 2^{-1/2}$ the image $\vartheta_\lambda[\mathcal{A}] = \{F \circ \vartheta_\lambda : F \in \mathcal{A}\}$ of $\mathcal{A}$ under ϑ_λ is dense in $L^2(\mathcal{G})$.*

PROOF. In essence, this proposition is based on [12, Theorem 3].

Since $F \circ \vartheta_\lambda = F$ on $\mathcal{G} \setminus \mathcal{C}$, it suffices to show that 1) for any $F \in \mathcal{A}$, $\lambda \mapsto F \circ \vartheta_\lambda$ is an $L^2(\mathcal{C})$-valued analytic function in the disc $|\lambda| < 2^{-1/2}$; 2) for any λ in this disc the image $\vartheta_\lambda[\mathcal{A}] = \{F \circ \vartheta_\lambda : F \in \mathcal{A}\}$ of $\mathcal{A}$ under ϑ_λ is dense in $L^2(\mathcal{C})$. Here the norm

$$\left(\int_{\mathcal{C}} |F(\zeta,\eta)|^2 \, d\zeta \, d\eta\right)^{1/2} = \left(\int_{\Pi} |F \circ \varkappa(x,y)|^2 \det \varkappa'(x,y) \, dx \, dy\right)^{1/2} \tag{5.2}$$

in $L^2(\mathcal{C})$ is equivalent to the norm

$$\left(\int_{\mathbb{R}_+} \|F \circ \varkappa(x,\cdot); L^2(\Omega)\|^2 \, dx\right)^{1/2} \tag{5.3}$$

in $L^2(\Pi)$, because $0 < \epsilon < \det \varkappa'(x,y) < 1/\epsilon$ uniformly in $(x,y) \in \Pi$ due to our assumptions on the diffeomorphism $\varkappa$.

1. We set $z = x + \lambda v(x)$. Then $|\Re z|^2 - |\Im z|^2 \ge x^2/2 - |\lambda|^2 |v(x)|^2$. Since $v(x) < x$, for all λ in the disc $|\lambda| \le \sqrt{1/2 - \epsilon}$, we get

$$|\Re z|^2 - |\Im z|^2 \ge \epsilon x^2 \ge \epsilon |v(x)|^2 \ge 2\epsilon |\Im z|^2.$$

Therefore $|\Im z| \le (1+2\epsilon)^{-1/2} \Re z$. On the other hand $\Re z \ge (1 - 2^{-1/2})x$. By the equality (5.1) with $f \in \mathscr{E}$, combined with the definition of the algebra $\mathscr{E}$, we conclude that the value $\|F \circ \vartheta_\lambda \circ \varkappa(x,\cdot); L^2(\Omega)\|$ decreases faster than any inverse power of x as $x \to +\infty$, uniformly in λ with $|\lambda| \le \sqrt{1/2 - \epsilon}$. It remains

to note that $(F \circ \vartheta_\lambda \circ \varkappa, G)_\Pi$ is analytic in $|\lambda| < 2^{-1/2}$ for any $G \in L^2(\Pi)$, where $(\cdot, \cdot)_\Pi$ stands for the inner product in $L^2(\Pi)$. The assertion 1 is proven.

2. Given $h \in C_0^\infty(\Pi)$ we will construct a sequence $f_\ell \in \mathscr{E}$, such that the function $\mathbb{R}_+ \ni x \mapsto g_\ell(x) = f_\ell(x + \lambda v(x)) \in C_0^\infty(\Omega)$ tends to h in $L^2(\Pi)$ as $\ell \to +\infty$. Since the set $C_0^\infty(\Pi)$ is dense in $L^2(\Pi)$, and the norms (5.2) and (5.3) are equivalent, this will imply that the set $\vartheta_\lambda[\mathcal{A}]$ is dense in $L^2(\mathcal{C})$.

Namely, let

$$f_\ell(z) = \sqrt{\frac{\ell}{\pi}} \int_{\mathbb{R}} h(x) \exp[-\ell(z - x - \lambda v(x))^2](1 + \lambda v'(x))\, dx, \ \ \ell \geq 1,$$

where $h(x) = 0$ for $x \leq 0$. It is clear that $z \mapsto f_\ell(z) \in C_0^\infty(\Omega)$ is an entire function. Since h is compactly supported, $z \mapsto \|f_\ell(z); L^2(\Omega)\|$ has the same falloff at infinity as $\exp(-\ell z^2)$, i.e. $f_\ell \in \mathscr{E}$. In order to prove that f_ℓ tends to h in $L^2(\Pi)$, we set

$$s(x, \tilde{x}; \lambda) = x + \lambda v(x) - \tilde{x} - \lambda v(\tilde{x}).$$

From the condition (2.3) on the scaling function v it follows that for all λ in the disc $|\lambda| \leq \sqrt{1/2 - \epsilon}$ we get $|\Re s|^2 - |\Im s|^2 \geq \epsilon|x - \tilde{x}|^2$, and therefore

$$|\exp(-s^2(x, \tilde{x}; \lambda))| \leq \exp(-\epsilon|x - \tilde{x}|^2). \tag{5.4}$$

For all real λ in the disc $|\lambda| \leq \sqrt{1/2 - \epsilon}$ we have

$$\sqrt{\frac{\ell}{\pi}} \int_{\mathbb{R}} \exp[-\ell(x + \lambda v(x) - \tilde{x} - \lambda v(\tilde{x}))^2](1 + \lambda v'(x))\, dx = \sqrt{\frac{\ell}{\pi}} \int_{\mathbb{R}} e^{-\ell s^2}\, ds = 1.$$

Due to (5.4) these equalities extend by analyticity to the disc $|\lambda| \leq \sqrt{1/2 - \epsilon}$. Thus we established the equality

$$h(\tilde{x}) - g_\ell(\tilde{x}) = \sqrt{\frac{\ell}{\pi}} \int_{\mathbb{R}} e^{-\ell s^2(x, \tilde{x}; \lambda)} (h(\tilde{x}) - h(x))(1 + \lambda v'(x))\, dx. \tag{5.5}$$

This together with (5.4) gives us the estimate

$$\|h(\tilde{x}) - g_\ell(\tilde{x}); L^2(\Omega)\| \leq C\sqrt{\ell} \int_{\mathbb{R}} e^{-\ell\varepsilon(x - \tilde{x})^2} \|h(\tilde{x}) - h(x); L^2(\Omega)\|\, dx. \tag{5.6}$$

It is known property of the Weierstraß singular integral [26] that for all $\tilde{x} \in \mathbb{R}$

$$\sqrt{\ell} \int_{\mathbb{R}} e^{-\ell\varepsilon(x - \tilde{x})^2} \|h(\tilde{x}) - h(x); L^2(\Omega)\|\, dx \to 0 \ \text{ as } \ \ell \to +\infty. \tag{5.7}$$

Due to (5.6) and (5.7) g_ℓ converges to h in the norm of $L^2(\Pi)$ as $\ell \to +\infty$.

6 Resolvent matrix elements meromorphic continuation

In this section we construct the meromorphic continuation in μ of the resolvent matrix elements $((\Delta-\mu)^{-1}F,G)$ with $F,G\in\mathcal{A}$, and complete the proof of Theorem 2.1.

Let us first obtain a relation between the matrix elements $((\Delta-\mu)^{-1}F,G)$ and some matrix elements of the resolvent $({}^{\lambda}\Delta-\mu)^{-1}$ with a real $\lambda\in\mathcal{D}_\alpha$.

For $F,G\in L^2(\mathcal{G})$ we define the sesquilinear form

$$(F,G)_\lambda=\int_{\mathcal{G}}F\cdot\overline{G}\sqrt{\det\mathsf{h}_\lambda}\,d\zeta\,d\eta,\quad \lambda\in\mathcal{D}_\alpha.$$

As the parameter R is sufficiently large, by the estimate (3.8) on h_λ^{-1} we have $0<c_1\le|\det\mathsf{h}_\lambda(\zeta,\eta)|\le c_2$ uniformly in $\lambda\in\mathcal{D}_\alpha$ and $(\zeta,\eta)\in\mathcal{G}$. Therefore the form $(\cdot,\cdot)_\lambda$ in $L^2(\mathcal{G})$ is continuous and nondegenerate, i.e. $|(F,G)_\lambda|\le C\|F;L^2(\mathcal{G})\|\|G;L^2(\mathcal{G})\|$, and for any nonzero $F\in L^2(\mathcal{G})$ there exists $G\in L^2(\mathcal{G})$, such that $(F,G)_\lambda\neq 0$.

Assume that $\lambda\in\mathcal{D}_\alpha$ is real. Then the form $(\cdot,\cdot)_\lambda$ is the inner product induced on $\mathcal{G}$ by the Riemannian metric h_λ, and $\sqrt{(\cdot,\cdot)_\lambda}$ is an equivalent norm in $L^2(\mathcal{G})$. The Laplace-Beltrami operator ${}^{\lambda}\Delta$ is a nonnegative selfadjoint operator in the space $L^2(\mathcal{G})$ endowed with the norm $\sqrt{(\cdot,\cdot)_\lambda}$, cf. Proposition 4.1.1. Thus the resolvent $({}^{\lambda}\Delta-\mu)^{-1}$ with $\mu<0$ is bounded. This allows to rewrite the equality $(\Delta-\mu)u=\big(({}^{\lambda}\Delta-\mu)(u\circ\vartheta_\lambda)\big)\circ\vartheta_\lambda^{-1}$ with $u\in C_0^\infty(\mathcal{G})$ in the form

$$(\Delta-\mu)^{-1}F=\big(({}^{\lambda}\Delta-\mu)^{-1}(F\circ\vartheta_\lambda)\big)\circ\vartheta_\lambda^{-1}. \tag{6.1}$$

Here F is in the set $\{F=(\Delta-\mu)u : u\in C_0^\infty(\mathcal{G})\}$. This set is dense in $L^2(\mathcal{G})$, because $C_0^\infty(\mathcal{G})$ is dense in $\mathring{H}^2(\mathcal{G})$, and the operator $\Delta-\mu:\mathring{H}^2(\mathcal{G})\to L^2(\mathcal{G})$ with $\mu<0$ realizes an isomorphism.

It is clear that $(F\circ\vartheta_\lambda,F\circ\vartheta_\lambda)_\lambda=(F,F)$. As a consequence, the (real) scaling $F\mapsto F\circ\vartheta_\lambda$ realizes an isomorphism in $L^2(\mathcal{G})$, and the equality (6.1) extends by continuity to all $F\in L^2(\mathcal{G})$. Taking the inner product of the equality (6.1) with $G\in L^2(\mathcal{G})$, and passing to the variables $(\tilde\zeta,\tilde\eta)=\vartheta_\lambda(\zeta,\eta)$ in the right hand side, we obtain the relation

$$\big((\Delta-\mu)^{-1}F,G\big)=\big(({}^{\lambda}\Delta-\mu)^{-1}(F\circ\vartheta_\lambda),G\circ\vartheta_{\overline{\lambda}}\big)_\lambda,\quad \lambda\in\mathcal{D}_\alpha\cap\mathbb{R}, \tag{6.2}$$

between the matrix elements of the resolvents $(\Delta-\mu)^{-1}$ and $({}^{\lambda}\Delta-\mu)^{-1}$.

We intend to implement the Aguilar-Balslev-Combes argument to the equality (6.2). In other words, for arbitrary $F,G\in\mathcal{A}$ and for some fixed $\mu<0$ we

will extend the equality (6.2) by analyticity to all λ in the disc $\mathcal{D}_\alpha$. Then the right hand side of (6.2) will provide the left hand side with a meromorphic continuation in μ across $\sigma_{ess}(\Delta)$.

By Proposition 5.1 $F \circ \vartheta_\lambda$ and $G \circ \vartheta_\lambda$ are analytic $L^2(\mathcal{G})$-valued functions of $\lambda \in \mathcal{D}_\alpha$. In order to implement the Aguilar-Balslev-Combes argument we need to show that the resolvent $({}^\lambda\Delta - \mu)^{-1}$ is an analytic function of λ. With this aim in mind we prove the following proposition.

Proposition 6.1 *The operator ${}^\lambda\Delta$ is m-sectorial with an independent of $\lambda \in \mathcal{D}_\alpha$ sector. Moreover, $\mathcal{D}_\alpha \ni \lambda \mapsto {}^\lambda\Delta$ is an analytic family in the sense of Kato [13].*

Remark 6.2 *Here m-sectorial means that the numerical range and the spectrum of ${}^\lambda\Delta$ are contained in some sector $\{\mu \in \mathbb{C} : |\arg(\mu - a)| \le b < \pi/2\}$.*

PROOF. The proof consists of two steps. On the first step we introduce an operator $[{}^\lambda\Delta]$ such that the difference ${}^\lambda\Delta - [{}^\lambda\Delta]$ is a first order differential operator. We estimate the numerical range of $[{}^\lambda\Delta]$, and show that the operator $[{}^\lambda\Delta]$ is m-sectorial. On the second step we prove that ${}^\lambda\Delta$ is m-sectorial.

Step one. We introduce the operator

$$[{}^\lambda\Delta] = \nabla_{\zeta\eta} \cdot \mathsf{h}_\lambda^{-1} \nabla_{\zeta\eta} \tag{6.3}$$

in the domain $\mathcal{G}$. From (2.6) and (6.3) it is clearly seen that ${}^\lambda\Delta - [{}^\lambda\Delta]$ is a first order operator. It is not hard to adapt the proof of Proposition 4.1 for the operator $[{}^\lambda\Delta]$. It turns out that $\sigma_{ess}({}^\lambda\Delta) = \sigma_{ess}([{}^\lambda\Delta])$, and the coercive estimate (4.5) with ${}^\lambda\Delta$ replaced by $[{}^\lambda\Delta]$ remains valid for all $\mu \notin \sigma_{ess}({}^\lambda\Delta)$. Therefore we can consider $[{}^\lambda\Delta]$ as a closed unbounded operator in $L^2(\mathcal{G})$ with the domain $\mathring{H}^2(\mathcal{G})$.

In order to estimate the numerical range of $[{}^\lambda\Delta]$, we establish the estimate

$$\left|\arg\Big([{}^\lambda\Delta]u, u\Big)\right| \le 2\alpha + \sigma < \pi/2, \quad \forall u \in \mathring{H}^2(\mathcal{G}), \tag{6.4}$$

for all $\lambda \in \mathcal{D}_\alpha$; here $(\cdot,\cdot)$ is the inner product in $L^2(\mathcal{G})$. It is clear that

$$\Big([{}^\lambda\Delta]u, u\Big) = \int_{\mathcal{G}} \Big\langle \mathsf{h}_\lambda^{-1} \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} u \Big\rangle \, d\zeta \, d\eta, \tag{6.5}$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product in $\mathbb{C}^{n+1}$.

Let us estimate the numerical range of the matrix $\mathsf{h}_\lambda^{-1}(\zeta, \eta)$. We shall rely on the estimate (3.8). Let $Z \in \mathbb{C}^{n+1}$. Observe that by virtue of the inequalities

$0 \leq v'(\zeta,\eta) \leq 1$ and $|\lambda| < \sin\alpha < 2^{-1/2}$ we have

$$\begin{aligned} &\Big| Z \cdot \operatorname{diag}\Big\{ \Big(1+\lambda v'(\zeta,\eta)\Big)^{-2}, \mathrm{Id} \Big\} \overline{Z} \Big| \geq |Z|^2/4, \\ &\Big| \arg\Big(Z \cdot \operatorname{diag}\Big\{ \Big(1+\lambda v'(\zeta,\eta)\Big)^{-2}, \mathrm{Id} \Big\} \overline{Z} \Big) \Big| < 2\alpha. \end{aligned} \tag{6.6}$$

Since R is sufficiently large, the constant $c(R)$ in (3.8) meets the estimate $4(n+1)^2 c(R) \leq \sin(\sigma/2)$ with some $\sigma \in (0, \pi/2 - 2\alpha)$. Then (3.8) together with (6.6) gives

$$\Big| \arg\Big(Z \cdot \mathsf{h}_\lambda^{-1}(\zeta,\eta) \overline{Z} \Big) \Big| \leq 2\alpha + \sigma < \pi/2, \quad \lambda \in \mathcal{D}_\alpha, \ (\zeta,\eta) \in \mathcal{G}.$$

Taking into account (6.5) we arrive at (6.4).

To the operator $[{}^\lambda\Delta]$ with $\lambda \in \mathcal{D}_\alpha$ we assign the sector $|\arg\mu| \leq 2\alpha + \sigma$ of angle less than π. Due to (6.4) the numerical range of $[{}^\lambda\Delta]$ is inside of its sector. Moreover, as we already found the essential spectrum $\sigma_{ess}([{}^\lambda\Delta]) = \sigma_{ess}({}^\lambda\Delta)$, we immediately see that it is in the sector of $[{}^\lambda\Delta]$, cf. Fig. 2.

Let μ be outside of the sector of $[{}^\lambda\Delta]$. Then the kernel of the operator $[{}^\lambda\Delta] - \mu$ is trivial. In order to see that $[{}^\lambda\Delta]$ is m-sectorial, it remains to show that the kernel of the adjoint operator $([{}^\lambda\Delta] - \mu)^*$ is trivial.

Since the symmetric matrix h_λ meets the equality $\overline{\mathsf{h}_\lambda} = \mathsf{h}_{\overline{\lambda}}$, we obtain the identity $([{}^\lambda\Delta]u, v) = (u, [{}^{\overline{\lambda}}\Delta]v)$ for all $u, v \in \mathring{H}^2(\mathcal{G})$, cf. (6.5). Hence the closed densely defined operator $[{}^{\overline{\lambda}}\Delta]$ is adjoint to $[{}^\lambda\Delta]$. Observe that $\overline{\mu}$ is outside of the sector of $[{}^{\overline{\lambda}}\Delta]$, provided that μ is outside of the sector of $[{}^\lambda\Delta]$. Therefore the kernel of $[{}^{\overline{\lambda}}\Delta] - \overline{\mu}$ is trivial for all μ outside of the sector of $[{}^\lambda\Delta]$, and the operator $[{}^\lambda\Delta]$ is m-sectorial.

Step two. Let us show that the operator ${}^\lambda\Delta - [{}^\lambda\Delta]$ has an arbitrarily small and uniform in $\lambda \in \mathcal{D}_\alpha$ relative bound with respect to $[{}^\lambda\Delta]$ in the operator sense.

From the estimate (3.8) on h_λ^{-1} and the definitions (2.6) and (6.3) of the operators ${}^\lambda\Delta$ and $[{}^\lambda\Delta]$, we see that coefficients $a^\lambda_{pq}(\zeta,\eta)$ of the first order differential operator ${}^\lambda\Delta - [{}^\lambda\Delta] = \sum_{p+|q|=1} a^\lambda_{pq} \partial^p_\zeta \partial^q_\eta$ are bounded uniformly in $(\zeta,\eta) \in \mathcal{G}$ and $\lambda \in \mathcal{D}_\alpha$. By a standard argument, based on the integration by parts, we conclude that the operator ${}^\lambda\Delta - [{}^\lambda\Delta]$ is uniformly bounded relative to Δ with an arbitrarily small relative bound; i.e. for all $\lambda \in \mathcal{D}_\alpha$ the estimate

$$\|({}^\lambda\Delta - [{}^\lambda\Delta])u; L^2(\mathcal{G})\| \leq \delta \|\Delta u; L^2(\mathcal{G})\| + c(\delta)\|u; L^2(\mathcal{G})\|, \quad \forall u \in \mathring{H}^2(\mathcal{G}), \tag{6.7}$$

holds with an arbitrarily small $\delta > 0$ and some $c(\delta)$, which depends only on δ.

In Lemma 6.3 below we establish the coercive estimate

$$\|u; \mathring{H}^2(\mathcal{G})\| \leq C(\|[{}^\lambda\Delta]u; L^2(\mathcal{G})\| + \|u; L^2(\mathcal{G})\|), \quad \forall u \in \mathring{H}^2(\mathcal{G}). \tag{6.8}$$

This estimate is far not that sharp as the estimate (4.5), however it holds uniformly in $\lambda \in \mathcal{D}_\alpha$. As a consequence of (4.1), (6.7), and (6.8) we obtain the estimate

$$\|({}^\lambda\Delta - [{}^\lambda\Delta])u; L^2(\mathcal{G})\| \leq \delta\|[{}^\lambda\Delta]u; L^2(\mathcal{G})\| + C(\delta)\|u; L^2(\mathcal{G})\|, \tag{6.9}$$

where $\delta > 0$ is arbitrarily small, and $C(\delta)$ depends only on δ.

The operator ${}^\lambda\Delta$ is m-sectorial as a perturbation with an arbitrarily small relative bound of the m-sectorial operator $[{}^\lambda\Delta]$, e.g. [13,23]. Moreover, the sector of ${}^\lambda\Delta$ is independent of λ, because the estimate (6.9) holds uniformly in λ, and the sector of $[{}^\lambda\Delta]$ is independent of λ.

The coefficients of the m-sectorial operator ${}^\lambda\Delta$ are analytic with respect to $\lambda \in \mathcal{D}_\alpha$. Therefore $\mathcal{D}_\alpha \ni \lambda \mapsto {}^\lambda\Delta$ is an analytic family of type A; i.e. for every $\lambda \in \mathcal{D}_\alpha$ the resolvent set of the closed unbounded operator ${}^\lambda\Delta$ in $L^2(\mathcal{G})$ is not empty, the domain $\mathring{H}^2(\mathcal{G})$ of ${}^\lambda\Delta$ does not depend on λ, and for any $u \in \mathring{H}^2(\mathcal{G})$ the function $\mathcal{D}_\alpha \ni \lambda \mapsto {}^\lambda\Delta u \in L^2(\mathcal{G})$ is analytic. As is known [13,23], an analytic family of type A is also analytic in the sense of Kato.

Lemma 6.3 *The coercive estimate* (6.8) *holds uniformly in* $\lambda \in \mathcal{D}_\alpha$.

PROOF. Consider the auxiliary operator

$$\mathsf{A}_\lambda = -\partial_\zeta\Big(1 + \lambda v'(\zeta, \eta)\Big)^{-2}\partial_\zeta - \partial^2_{\eta_1} - \cdots - \partial^2_{\eta_n}.$$

By the estimate (3.8) the coefficients of the differential operator $\mathsf{A}_\lambda - [{}^\lambda\Delta]$ can be made arbitrarily small (uniformly in $\lambda \in \mathcal{D}_\alpha$) by taking a sufficiently large R. Therefore for any $\epsilon > 0$ there exists $R > 0$, such that

$$\|\mathsf{A}_\lambda u - [{}^\lambda\Delta]u; L^2(\mathcal{G})\|^2 \leq \epsilon\|u; \mathring{H}^2(\mathcal{G})\|^2. \tag{6.10}$$

Below we will prove the uniform in λ estimate

$$\|\Delta u; L^2(\mathcal{G})\|^2 \leq C(\|\mathsf{A}_\lambda u; L^2(\mathcal{G})\|^2 + \|u; L^2(\mathcal{G})\|^2) \tag{6.11}$$

with some independent of R and u constant C. Then the coercive stimate (6.8) will follow from (6.10), (6.11), and (4.1).

Let us prove (6.11) for $u \in C_0^\infty(\mathcal{G})$. From the inequalities $0 \leq v'(\zeta, \eta) \leq 1$, $|\lambda| < \sin\alpha$, and $\alpha < \pi/4$, it is easily seen that

$$\Re(\mathsf{A}_\lambda u, u) = \Re\Big((1 + \lambda v')^{-2}\partial_\zeta u, \partial_\zeta u\Big) + \sum\Big(\partial_{\eta_j} u, \partial_{\eta_j} u\Big) \geq (\Delta u, u)/c$$

with $1/c = (\cos 2\alpha)/4 > 0$. As a consequence we have

$$\begin{aligned} \|\Delta u; L^2(\mathcal{G})\|^2 = (\Delta \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} u) \leq c|(\mathsf{A}_\lambda \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} u)| \\ \leq c|(\nabla_{\zeta\eta} \mathsf{A}_\lambda u, \nabla_{\zeta\eta} u)| + c|([\mathsf{A}_\lambda, \nabla_{\zeta\eta}] u, \nabla_{\zeta\eta} u)|. \end{aligned} \tag{6.12}$$

In the first term of the right hand side of (6.12) we use that

$$|(\nabla_{\zeta\eta} \mathsf{A}_\lambda u, \nabla_{\zeta\eta} u)| = |(\mathsf{A}_\lambda u, \Delta u)| \leq \delta \|\Delta u; L^2(\mathcal{G})\|^2 + \delta^{-1} \|\mathsf{A}_\lambda u; L^2(\mathcal{G})\|^2 \tag{6.13}$$

for an arbitrarily small $\delta > 0$. The commutator $[\mathsf{A}_\lambda, \nabla_{\zeta\eta}]$ in the second term is a second order operator satisfying the estimate

$$\|[\mathsf{A}_\lambda, \nabla_{\zeta\eta}] u; L^2(\mathcal{G})\| \leq C(\|\Delta u; L^2(\mathcal{G})\| + \|\nabla_{\zeta\eta} u; L^2(\mathcal{G})\|).$$

As a consequence, for an arbitrarily small $\delta > 0$ we get

$$\begin{aligned} &|([\mathsf{A}_\lambda, \nabla_{\zeta\eta}] u, \nabla_{\zeta\eta} u)| \leq C(\|\Delta u; L^2(\mathcal{G})\| + \|\nabla_{\zeta\eta} u; L^2(\mathcal{G})\|)\|\nabla_{\zeta\eta} u; L^2(\mathcal{G})\| \\ &\leq \delta \|\Delta u; L^2(\mathcal{G})\|^2 + c(\delta)(\nabla_{\zeta\eta} u, \nabla_{\zeta\eta} u) \leq 2\delta \|\Delta u; L^2(\mathcal{G})\|^2 + C(\delta)\|u; L^2(\mathcal{G})\|^2. \end{aligned} \tag{6.14}$$

Now for all $u \in C_0^\infty(\mathcal{G})$ the estimate (6.11) is readily apparent from (6.12), (6.13), and (6.14). By continuity it extends to all $u \in \mathring{H}^2(\mathcal{G})$

Now we are in position to prove Theorem 2.1.

Proof of Theorem 2.1 The assertion *2* is a direct consequence of Proposition 4.1.

3. Let $\lambda \in \mathcal{D}_\alpha$ be fixed. By Proposition 6.1 there is a point $\mu < 0$ in the resolvent set of ${}^\lambda\Delta$. Every $\mu < 0$ is in the simply connected set $\mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$, cf. Fig. 2. This implies that ${}^\lambda\Delta - \mu : \mathring{H}^2(\mathcal{G}) \to L^2(\mathcal{G})$ is an analytic Fredholm operator function of $\mu \in \mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$. It is known e.g. [14, Proposition A.8.4], the spectrum of an analytic Fredholm operator function consists of isolated eigenvalues of finite algebraic multiplicity. Thus $\sigma({}^\lambda\Delta) = \sigma_{ess}({}^\lambda\Delta) \cup \sigma_d({}^\lambda\Delta)$.

5. Recall that for all real $\lambda \in \mathcal{D}_\alpha$ and $F, G \in L^2(\mathcal{G})$ the equality (6.2) is valid. To this equality we implement a variant of the Aguilar-Balslev-Combes argument. Namely, let F and G be in the set $\mathcal{A}$ of partial analytic vectors. Then by Proposition 5.1 $F \circ \vartheta_\lambda$ and $G \circ \vartheta_\lambda$ are $L^2(\mathcal{G})$-valued analytic functions of λ in the disc $\mathcal{D}_\alpha$. In order to see that the equality (6.2) extends by analyticity to all $\lambda \in \mathcal{D}_\alpha$, we consider the resolvent $({}^\lambda\Delta - \mu)^{-1}$. By Proposition 6.1 $\lambda \mapsto {}^\lambda\Delta$ is an analytic family in the sense of Kato, hence the resolvent is an analytic function of the variables μ and λ on the set $\{(\mu, \lambda) \in \mathbb{C} \times \mathcal{D}_\alpha : \mu \notin \sigma({}^\lambda\Delta)\}$, e.g. [23, Theorem XII.7]. Let μ be a negative number outside of the sector of the m-sectorial operator ${}^\lambda\Delta$, cf. Proposition 6.1. Then $\mu \notin \sigma({}^\lambda\Delta)$ for all

$\lambda \in \mathcal{D}_\alpha$, and the resolvent $\mathcal{D}_\alpha \ni \lambda \mapsto ({}^\lambda\Delta - \mu)^{-1}$ is an analytic function. As a consequence, the equality (6.2) extends by analyticity to all $\lambda \in \mathcal{D}_\alpha$.

We take some $\lambda \in \mathcal{D}_\alpha$ with $\Im\lambda \neq 0$, and consider the Left Hand Side (LHS) and the Right Hand Side (RHS) of the equality (6.2) as functions of μ. The LHS is meromorphic in $\mu \in \mathbb{C} \setminus \sigma_{ess}(\Delta)$ with poles at the points of $\sigma_d(\Delta)$. Note that the spectrum $\sigma(\Delta)$ is a subset of the half-line $\mathbb{R}_+$ as Δ is a positive selfadjoint operator. On the other hand, the RHS is a meromorphic function on the set $\mu \in \mathbb{C} \setminus \sigma_{ess}({}^\lambda\Delta)$. Therefore the RHS provides the LHS with a meromorphic continuation in μ from $\mathbb{C} \setminus \sigma_{ess}(\Delta)$ across the intervals (ν_j, ν_{j+1}), $j \in \mathbb{N}$, between the thresholds to the strips between the rays of $\sigma_{ess}({}^\lambda\Delta)$, cf. Fig. 2.

It is clear that the meromorphic continuation can have poles only at points of $\sigma_d({}^\lambda\Delta)$. Conversely, let $\mu_0 \in \sigma_d({}^\lambda\Delta)$, and let P be the corresponding Riesz projection (i.e. the first order residue of $({}^\lambda\Delta - \mu)^{-1}$ at the pole μ_0). The kernel $\ker({}^\lambda\Delta - \mu_0) \neq \{0\}$ is in the range of P. Recall that the form $(\cdot,\cdot)_\lambda$ in $L^2(\mathcal{G})$ is nondegenerate, and by Proposition 5.1 the sets $\vartheta_\lambda[\mathcal{A}]$ and $\vartheta_{\overline{\lambda}}[\mathcal{A}]$ are dense in $L^2(\mathcal{G})$. Therefore for some $F, G \in \mathcal{A}$ we must have $\left(\mathsf{P}F \circ \vartheta_\lambda, G \circ \vartheta_{\overline{\lambda}}\right)_\lambda \neq 0$. Thus μ_0 is a pole.

1. The LHS of the equality (6.2) is an independent of v analytic function of $\mu \in \mathbb{C} \setminus \sigma(\Delta)$. Hence the meromorphic continuation of the LHS and its poles are independent of the scaling function v. This together with the assertion *5* implies that $\sigma_d({}^\lambda\Delta)$ is independent of v. By the proven assertions *2* and *3* the essential spectrum $\sigma_{ess}({}^\lambda\Delta)$ is independent of v and $\sigma({}^\lambda\Delta) = \sigma_{ess}({}^\lambda\Delta) \cup \sigma_d({}^\lambda\Delta)$.

4. Let $\mu \in \sigma_d({}^\lambda\Delta)$ be fixed. As λ changes continuously in the disc $\mathcal{D}_\alpha$ and $\mu \notin \sigma_{ess}({}^\lambda\Delta)$, the RHS of (6.2) provides the LHS with one and the same meromorphic continuation to a neighborhood of μ. Therefore μ remains to be a pole of the meromorphic continuation. Then $\mu \in \sigma_d({}^\lambda\Delta)$ by the assertion *5*.

6. Let λ be a non-real number in the disc $\mathcal{D}_\alpha$. Consider the projection

$$\mathsf{P} = \operatorname*{s-lim}_{\epsilon \downarrow 0} i\epsilon(\Delta - \mu_0 - i\epsilon)^{-1}$$

onto the eigenspace of the selfadjoint operator Δ. Suppose that $\mu_0 \in \mathbb{R} \setminus \sigma({}^\lambda\Delta)$ (then $\mu_0 \neq \nu_j$ for all $j \in \mathbb{N}$). Then for any $F, G \in \mathcal{A}$ the RHS of (6.2) is an analytic function of μ in a neighborhood of μ_0. The equality (6.2) implies that $(\mathsf{P}F, G) = 0$. By Proposition 5.1 the set $\mathcal{A}$ is dense in $L^2(\mathcal{G})$, and hence $\mathsf{P} = 0$. Thus $\ker(\Delta - \mu_0) = \{0\}$.

Now we assume that $\mu_0 \in \mathbb{R}$ and $\mu_0 \in \sigma_d({}^\lambda\Delta)$. Then the resolvent $({}^\lambda\Delta - \mu)^{-1}$ has a pole at μ_0. The sets $\vartheta_\lambda[\mathcal{A}]$ and $\vartheta_{\overline{\lambda}}[\mathcal{A}]$ are dense in $L^2(\mathcal{G})$. Hence there exist $F, G \in \mathcal{A}$, such that μ_0 is a pole for the RHS of (6.2). The equality (6.2) implies that $(\mathsf{P}F, G) \neq 0$, and thus $\ker(\Delta - \mu_0) \neq \{0\}$.

7. The RHS of (6.2) with $\Im\lambda > 0$, and therefore the LHS, being defined on the dense subset $\mathcal{A}$ of $L^2(\mathcal{G})$, has limits at the points $\mathbb{R} \setminus \sigma(^\lambda\Delta)$ as μ tends to the real line from $\mathbb{C}^+$. Since the set $\mathbb{R} \cap \sigma(^\lambda\Delta)$ is countable, the Dirichlet Laplacian Δ has no singular continuous spectrum, e.g. [23, Theorem XII.20].$\square$

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